Simplical Sets

Δ = category of simplicial operators

$$= \begin{cases} \text{obj}: & [n] := \{0<1<2<\cdots<n\}, \quad n \geq 0 \\ \text{mor} & f:[m]\to[n]: \text{ order preserving maps} \end{cases}$$

$[n]$ is also a category: $\begin{cases} \text{obj}: & 0,1,2,\dots,n \\ \text{mor } p\to q: & \exists, \text{unique if } p\leq q \\ & \text{not exist if } p>q \end{cases}$

maps in Δ $\Longleftrightarrow$ functors

$[4]$:

$$\begin{tikzcd}
0 \arrow[loop above] \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend right] \arrow[rrrr, bend right=40] & 1 \arrow[loop above] \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend right] & 2 \arrow[loop above] \arrow[r] \arrow[rr, bend left] & 3 \arrow[loop above] \arrow[r] & 4 \arrow[loop above]
\end{tikzcd}$$

Notation: $f = \langle f_0 f_1 \cdots f_n \rangle : [n] \longrightarrow [m]$ $\leftarrow$

where $0 \leq f_0 \leq f_1 \leq \cdots \leq f_n \leq m$

so $f(k) = f_k$

Rem: $d^i : [n-1] \rightarrowtail [n]$, "face" $i = 0,\dots,n$

$s^i : [n+1] \twoheadrightarrow [n]$ $i = 0,\dots,n$ "degeneracy"

Simplicial set: functor $X : \Delta^{op} \to Set$

so: each $n \geq 0 \rightsquigarrow X([n]) = X_n$ set

set of ~~n-simplices~~ n-cells.

each $f : [m] \to [n]$,

$$\begin{array}{ccc} X_n & \xrightarrow{f^*} & X_m \\ \cup & & \cup \\ a & \longmapsto & af \end{array}$$

functor

such that $a \cdot 1d_{[m]} = a$

and $(af)g = a(fg)$

$$[m] \xrightarrow{g} [n] \xrightarrow{f} [p], \quad fg : [m] \to [p]$$

Notation: $f = \langle f_0 f_1 \cdots f_n \rangle : [n] \to [m]$

write $af = a_{f_0 f_1 \cdots f_n}$

$a \in X_m$

eg.: $\langle i \rangle : [0] \to [n]$ $\qquad a \in X_n$

$a_i := a\langle i \rangle \in X_0$

vertices

$\langle ij \rangle : [1] \to [n]$ $\qquad a \in X_n$

$a_{ij} = a\langle ij \rangle \in X_1$

$0 \leq i \leq j \leq n$

edges

sSet = cat of simplicial sets

$$= \begin{cases} \underline{\text{obj}}: & X : \Delta^{op} \to \text{Set} \quad (\text{simplicial sets} = \text{functors}) \\ \underline{\text{mor}}: & X \xrightarrow{\varphi} Y \quad \text{nat transf of functs} \end{cases}$$

ie, $\varphi_n : X_n \to Y_n$ functions, $n \geq 0$

such that $(\varphi a) f = \varphi(af)$

$\forall\ a \in X_n, \quad f : [m] \to [n]$ in Δ

Ex: $\underline{\text{Discrete simplicial set}}$: $X \in$ sSet

such that all $f^* : X_n \to X_m$ are $\underline{\text{isos}}$.

$\underline{\text{Ex}}$: S set $\Longrightarrow$ $S^{disc} \in$ sSet

so $(S^{disc})_n = S$, and all $f^* = \mathrm{id}_S$

$\underline{\text{Fact}}$: Every discrete simplicial set is isomorphic to an S^{disc}

$(X$ discrete s.s., $\Rightarrow X \approx (X_0)^{disc})$.

$\underline{\text{Functor}}$: $\text{Set} \to \text{sSet}$ is fully faithful.

$S \longmapsto S^{disc}$

ie $\quad S, T \in \mathrm{Set}$

$$\mathrm{Hom}_{\mathrm{Set}}(S,T) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{sSet}}(S^{disc}, T^{disc})$$

is a bijection

$$\mathrm{Set} \hookrightarrow \mathrm{sSet}$$

abuse language: identify a set w/ corr. discrete simplicial set

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<u>Standard n-simplex</u>: Δ^n $\quad (\Delta[n])$

$$\Delta^n : \mathrm{Hom}_{\Delta}(-, [n]) : \Delta^{op} \to \mathrm{Set}$$

ie, $(\Delta^n)_m = \{ [m] \xrightarrow{a} [n] \in \Delta \}$

given $[m'] \xrightarrow{f} [m] \implies (\Delta^n)_m \xrightarrow{f^*} (\Delta^n)_{m'}$

$$a \longmapsto a \circ f$$

<u>generator</u>: $\iota_n = \langle 01 \cdots n \rangle \in (\Delta^n)_n$

$$\overset{\shortparallel}{id_{[n]}}$$

<u>Yoneda Lemma</u>: <u>bijection</u>: $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \xrightarrow{\sim} X_n$

$$g \longmapsto g(\iota_n)$$

ie, $\forall \ a \in X_n \implies \exists$ unique $f_a : \Delta^n \to X$ (represents map)

st $f_a(1_n) = a$

abuse of

Notation: given $a \in X_n$, also write $a : \Delta^n \to X$ for the representing map

Note: $f : [m] \to [n] \implies \Delta^m \xrightarrow{\Delta^f} \Delta^n$ map of sSets

$$\mathrm{Hom}(-,[m]) \to \mathrm{Hom}(-,[n])$$

abuse of notation: $a \in (\Delta^m)_p \longmapsto \Delta^f(a) \in (\Delta^n)_p$

$f : [m] \to [n]$ $\quad$ $\Vert$ $\quad$ "$\Vert$ fa"

$a : [p] \to [m] \implies fa : [p] \to [n]$

Ex: $(\Delta^0)_n = \{[n] \xrightarrow{\langle 0 \rangle} [0]\}$ $\quad \Delta^0 = * \cong *^{disc}$

terminal object in sSet

$\emptyset \in sSet$ $\quad \emptyset_n := \emptyset$ $\quad$ (really $\emptyset^{disc}$)

initial object in sSet

S = totally ordered set $\Longrightarrow$ $\Delta^S \in sSet$

with $(\Delta^S)_k := \left\{ \begin{array}{c} \text{order pres.} \\ [k] \longrightarrow S \end{array} \right\}$

If $0 < |S| < \infty$, then $S \underset{\substack{\uparrow \\ \text{unique iso.}}}{\cong} \Delta^n$ $\quad 1+n = |S|$

Ex: If $S \subseteq [n] \implies \Delta^S \subseteq \Delta^n$ "subcomplex" $\in sSet$

<u>Important:</u>

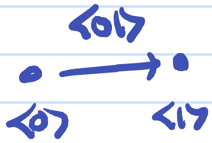
$$\begin{tikzcd} f : [m] \arrow[rr] \arrow[dr, two heads, "f_{surj}"'] & & {[n]} \\ & S \arrow[ur, hook, "f_{inj}"'] & \end{tikzcd} \qquad S = f([m]) \subseteq [n]$$

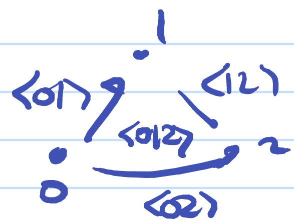
$$\Longrightarrow \quad \begin{tikzcd} \Delta^m \arrow[rr, "\Delta^f"] \arrow[dr, two heads, "\Delta^{f_{surj}}"'] & & \Delta^n \\ & \Delta^S \arrow[ur, hook, "\Delta^{f_{inj}}"'] & \end{tikzcd}$$

$$\Delta^S \cong \Delta^k$$

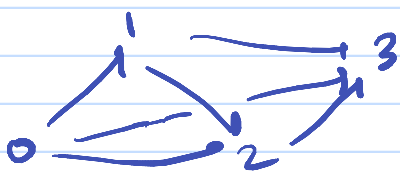
Pictures:

Δ^0 $\qquad\qquad$ Δ^1 $\qquad\qquad$ Δ^2





Δ^3



$\Rightarrow$

$$(\Delta^1)_1 = \{\langle 00\rangle, \langle 01\rangle, \langle 11\rangle\}$$

$$(\Delta^2)_1 = \{\langle 00\rangle, \dots, \langle 22\rangle\}$$

$$(\Delta^2)_2 = \{\langle 000\rangle, \langle 001\rangle, \dots, \dots, \langle 222\rangle\}$$

$$\Delta^+ \supseteq \Delta$$

cosimplicial set $\quad \Delta \to Set$

Nerve of a category:

C - category $\Longrightarrow$ $NC \in sSet$

$$(NC)_n := Hom_{Cat}([n], C) = \left\{ [n] \xrightarrow{f} C \text{ functor} \right\}$$

$$g : [m] \to [n] \Longrightarrow \begin{array}{ccc} (NC)_n & \longrightarrow & (NC)_m \\ f & \longmapsto & fg \end{array}$$

Ex: $N[n] = \Delta^n$

Ex: $F : C \to D$ functor of cats

$\implies NF : NC \to ND \in sSet$

Note: $(NC)_0 = \{\text{objects of } C\}$

$(NC)_1 = \{\text{morphism of } C\}$

$$\begin{tikzcd}
(NC)_1 \arrow[r, shift left, "\langle 0 \rangle^*"] \arrow[r, shift right, "\langle 1 \rangle^*"'] & (NC)_0 \arrow[l, bend left, "\langle 00 \rangle^*"]
\end{tikzcd}$$

$(a \xrightarrow{f} b) \mapsto a \quad \langle 0 \rangle^*$

$\phantom{(a \xrightarrow{f} b)} \mapsto b \quad \langle 1 \rangle^*$

$[1] \xrightarrow{\langle 00 \rangle} [0]$

$0 \mapsto 0$

$1 \mapsto 0$

$(NC)_0 \to (NC)_1$

$a \mapsto 1_a$

a object $\qquad 1_a$ id. morphism of a.

$(NC)_2 = \{(f,g),\ f,g \in \text{mor}\, C \text{ st } gf \text{ is defined}\}$

$= \{(0 \to 1 \to 2) \to C\}$

$$\begin{tikzcd}
 & f \in (NC)_1 & \\
 & (f,g) \in (NC)_2 \arrow[u, "\langle 01 \rangle^*"] \arrow[dl, "\langle 02 \rangle^*"'] \arrow[dr, "\langle 12 \rangle^*"] & \\
gf \in (NC)_1 & & g \in (NC)_1
\end{tikzcd}$$

you can reconstruct C from its nerve

Prop: $(NC)_n := \{(g_1, g_2, \ldots, g_n) \mid g_i \in \mathrm{mor}\, C,\ \mathrm{target}(g_{i-1}) = \mathrm{source}(g_i)\}$

Prop: $X \in sSet$ is isomorphic to NC for some some cat C iff

$\forall\, n \geq 2$,

$$X_n \xrightarrow{=} \{(g_1, \ldots, g_n) \in X_1^{\times n} \mid g_{i-1}\langle 1\rangle = g_i\langle 0\rangle\}$$

$$a \longmapsto (a\langle 01\rangle, a\langle 12\rangle, a\langle 23\rangle, \ldots, a\langle n-1, n\rangle)$$

is a bijection

$X_n \subseteq X_1 \times_{X_0} \cdots \times_{X_0} X_1$

Idea: $X \in sSet$ like this

Prop: $N : Cat \hookrightarrow sSet$ is fully faithfull

(categories, functors)

i.e., $\mathrm{Hom}_{Cat}(C, D) \xrightarrow[\cong]{N} \mathrm{Hom}_{sSet}(NC, ND)$

$\{\text{functors } C \to D\}$

Abuse of notation: "identify" categories with simplicial sets.